Faltung der Intenstätsverteilung hinter einer Lochblende mit einer Rechteckfunktion

Definition der Intensitätsverteilung

In[1]:=
```
ClearAll[f, I0, x0, x, Ib, G];
f[x_] = I0 * (2 BesselJ[1, G * (x - x0)] / (G * (x - x0)))^2 + Ib
```

Out[2]=
$$Ib + \frac{4\ I0\ \text{BesselJ}[1,\ G\ (x - x0)]^2}{G^2\ (x - x0)^2}$$

In[3]:=
```
Limit[f[x], x → x0]
```

Out[3]=
$$I0 + Ib$$

In[4]:=
```
TeXForm[f[x]]
```

Out[4]//TeXForm=
```
\frac{4 \text{I0} J_1(G (x-\text{x0}))}{}^2}{G^2
   (x-\text{x0})^2}+\text{Ib}
```

Definition der Rechteckfunktion mit der Breite 2b

In[5]:=
```
ClearAll[r, b, x];
r[x_] = UnitBox[x / (2 b)] / (2 b)
```

Out[6]=
$$\frac{\text{UnitBox}\left[\frac{x}{2\,b}\right]}{2\,b}$$

In[7]:=
```
ClearAll[b]; Integrate[r[x], {x, -Infinity, Infinity}, Assumptions → {b > 0}]
```

Out[7]=
$$1$$

In[8]:=
```
TeXForm[r[x]]
```

Out[8]//TeXForm=
```
\frac{\Pi \left(\frac{x}{2 b}\right)}{2 b}
```

In[9]:=
```
PiecewiseExpand[r[x]]
```

Out[9]=
$$\begin{cases} \frac{1}{2\,b} & -1 \le \frac{x}{b} \le 1 \\ 0 & \text{True} \end{cases}$$

In[10]:=
```
TeXForm[PiecewiseExpand[r[x]]]
```

Out[10]//TeXForm=
```
\begin{cases}
 \frac{1}{2 b} & -1\leq \frac{x}{b}\leq 1 \\
 0 & \text{True}
\end{cases}
```

Ausnutzung der Distributivität der Faltung
r * (f + h) = r * f + r * h
und der Assoziativität mit der skalaren Multiplikation
r * (a f) = a (r * f)
sowie ersetzen von (x - x0) durch x führt zu:

In[11]:=
```
f[x_] = (2 BesselJ[1, G * x] / (G * x)) ^ 2
```

Out[11]=
$$\frac{4\ \text{BesselJ}[1,\ G\,x]^2}{G^2\,x^2}$$

In[12]:=
```
h[x_] = 1
```

Out[12]= 1

In[13]:=
```
Convolve[ r[y], h[y] , y, x, Assumptions → b > 0]
```

Out[13]= 1

In[14]:=
```
Imsp = Convolve[r[y], f[y], y, x, Assumptions → b > 0]
```

Out[14]=
$$\Big(2\,\Big(2\,b^3\,G^2\,\text{BesselJ}[0,\ G\,(b-x)\,]^2 - 2\,b^2\,G^2\,x\,\text{BesselJ}[0,\ G\,(b-x)\,]^2 - 2\,b\,G^2\,x^2\,\text{BesselJ}[0,\ G\,(b-x)\,]^2 + 2\,G^2\,x^3\,\text{BesselJ}[0,\ G\,(b-x)\,]^2 + 2\,b^3\,G^2\,\text{BesselJ}[0,\ G\,(b+x)\,]^2 + 2\,b^2\,G^2\,x\,\text{BesselJ}[0,\ G\,(b+x)\,]^2 - 2\,b\,G^2\,x^2\,\text{BesselJ}[0,\ G\,(b+x)\,]^2 - 2\,G^2\,x^3\,\text{BesselJ}[0,\ G\,(b+x)\,]^2 - 2\,b^2\,G\,\text{BesselJ}[0,\ G\,(b-x)\,]\,\text{BesselJ}[1,\ G\,(b-x)\,] + 2\,G\,x^2\,\text{BesselJ}[0,\ G\,(b-x)\,]\,\text{BesselJ}[1,\ G\,(b-x)\,] - b\,\text{BesselJ}[1,\ G\,(b-x)\,]^2 + 2\,b^3\,G^2\,\text{BesselJ}[1,\ G\,(b-x)\,]^2 - x\,\text{BesselJ}[1,\ G\,(b-x)\,]^2 - 2\,b^2\,G^2\,x\,\text{BesselJ}[1,\ G\,(b-x)\,]^2 - 2\,b\,G^2\,x^2\,\text{BesselJ}[1,\ G\,(b-x)\,]^2 + 2\,G^2\,x^3\,\text{BesselJ}[1,\ G\,(b-x)\,]^2 - 2\,b^2\,G\,\text{BesselJ}[0,\ G\,(b+x)\,]\,\text{BesselJ}[1,\ G\,(b+x)\,] + 2\,G\,x^2\,\text{BesselJ}[0,\ G\,(b+x)\,]\,\text{BesselJ}[1,\ G\,(b+x)\,] - b\,\text{BesselJ}[1,\ G\,(b+x)\,]^2 + 2\,b^3\,G^2\,\text{BesselJ}[1,\ G\,(b+x)\,]^2 + x\,\text{BesselJ}[1,\ G\,(b+x)\,]^2 + 2\,b^2\,G^2\,x\,\text{BesselJ}[1,\ G\,(b+x)\,]^2 - 2\,b\,G^2\,x^2\,\text{BesselJ}[1,\ G\,(b+x)\,]^2 - 2\,G^2\,x^3\,\text{BesselJ}[1,\ G\,(b+x)\,]^2\Big)\Big) \Big/ \Big(3\,b\,G^2\,(b-x)\,(b+x)\Big)$$

In[15]:=
```
Imsp1 = FullSimplify[Imsp]
```

Out[15]=
$$\frac{1}{3\,b}2\left(2\,(b-x)\,\text{BesselJ}[0,\ G\,(b-x)\,]^2 + 2\,(b+x)\,\text{BesselJ}[0,\ G\,(b+x)\,]^2 - \frac{2\,\text{BesselJ}[0,\ G\,(b-x)\,]\,\text{BesselJ}[1,\ G\,(b-x)\,]}{G} - \frac{2\,\text{BesselJ}[0,\ G\,(b+x)\,]\,\text{BesselJ}[1,\ G\,(b+x)\,]}{G} + \frac{\frac{\left(-1+2\,G^2\,(b-x)^2\right)\,\text{BesselJ}[1,G\,(b-x)\,]^2}{b-x} + \frac{\left(-1+2\,G^2\,(b+x)^2\right)\,\text{BesselJ}[1,G\,(b+x)\,]^2}{b+x}}{G^2}\right)$$

In[16]:=
```
TeXForm[Imsp1]
```

Out[16]//TeXForm=
```
\frac{2 \left(\frac{\frac{\left(2 G^2 (b-x)^2-1\right) J_1(G
   (b-x)){}^2}{b-x}+\frac{\left(2 G^2 (b+x)^2-1\right) J_1(G
   (b+x)){}^2}{b+x}}{G^2}+2 (b-x) J_0(G (b-x)){}^2-\frac{2 J_1(G
   (b-x)) J_0(G (b-x))}{G}+2 (b+x) J_0(G (b+x)){}^2-\frac{2 J_0(G
   (b+x)) J_1(G (b+x))}{G}\right)}{3 b}
```

In[17]:=
```
Imsp1f = Function[x, Evaluate[Imsp1]]
```

Out[17]=
$$\text{Function}\left[x, \frac{1}{3\,b} 2 \left(2\,(b-x)\,\text{BesselJ}[0, G\,(b-x)]^2 + 2\,(b+x)\,\text{BesselJ}[0, G\,(b+x)]^2 - \frac{2\,\text{BesselJ}[0, G\,(b-x)]\,\text{BesselJ}[1, G\,(b-x)]}{G} - \frac{2\,\text{BesselJ}[0, G\,(b+x)]\,\text{BesselJ}[1, G\,(b+x)]}{G} + \frac{\frac{\left(-1+2\,G^2\,(b-x)^2\right)\,\text{BesselJ}[1,G\,(b-x)]^2}{b-x} + \frac{\left(-1+2\,G^2\,(b+x)^2\right)\,\text{BesselJ}[1,G\,(b+x)]^2}{b+x}}{G^2}\right)\right]$$

In[18]:=
```
IxO = Imsp1f[0]
```

Out[18]=
$$\frac{1}{3\,b} 2 \left(4\,b\,\text{BesselJ}[0, b\,G]^2 - \frac{4\,\text{BesselJ}[0, b\,G]\,\text{BesselJ}[1, b\,G]}{G} + \frac{2\,\left(-1+2\,b^2\,G^2\right)\,\text{BesselJ}[1, b\,G]^2}{b\,G^2}\right)$$

In[19]:=
```
TeXForm[IxO]
```

Out[19]//TeXForm=
```
\frac{2 \left(\frac{2 \left(2 b^2 G^2-1\right) J_1(b G){}^2}{b
   G^2}+4 b J_0(b G){}^2-\frac{4 J_1(b G) J_0(b G)}{G}\right)}{3 b}
```

In[20]:=
```
Limit[IxO, b → 0]
```

Out[20]=
$$1$$

In[21]:=
```
Imsp1b0 = Limit[Imsp1f[x], b → 0]
```

Out[21]=
$$-\frac{1}{3\,G^2\,x^2} 4 \left(-2\,G^2\,x^2\,\text{BesselJ}[0, G\,x]^2 + 2\,G\,x\,\text{BesselJ}[0, G\,x]\left(\left(1+2\,G^2\,x^2\right)\text{BesselJ}[1, G\,x] - G\,x\,\text{BesselJ}[2, G\,x]\right) + \text{BesselJ}[1, G\,x]\left(\left(1-8\,G^2\,x^2\right)\text{BesselJ}[1, G\,x] + 2\,G\,x\left(-1+2\,G^2\,x^2\right)\text{BesselJ}[2, G\,x]\right)\right)$$

In[22]:=
```
FullSimplify[Imsp1b0]
```

Out[22]=
$$\text{Hypergeometric0F1Regularized}\left[2, -\frac{1}{4} G^2\,x^2\right]^2$$

Numerischer Test

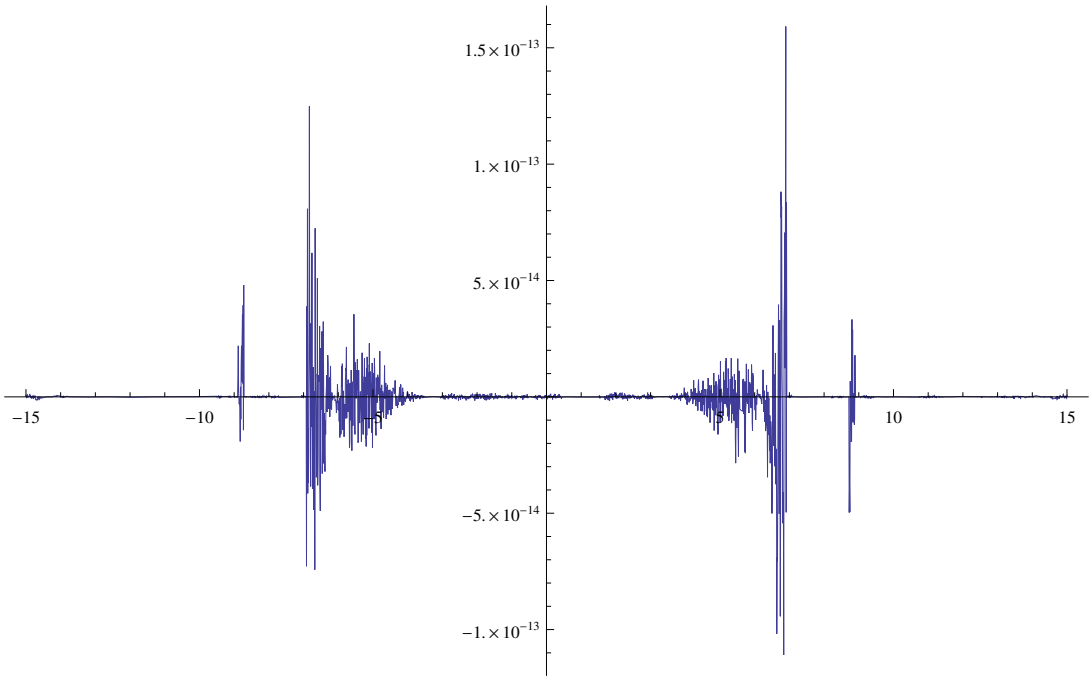
In[23]:=
```
Imsp1b0f = Function[x, Evaluate[Imsp1b0]]
```

Out[23]=
$$\text{Function}\left[x, -\frac{1}{3\,G^2\,x^2}4\,\left(-2\,G^2\,x^2\,\text{BesselJ}[0,\,G\,x]^2+2\,G\,x\,\text{BesselJ}[0,\,G\,x]\right.\right.$$
$$\left.\left(\left(1+2\,G^2\,x^2\right)\,\text{BesselJ}[1,\,G\,x]-G\,x\,\text{BesselJ}[2,\,G\,x]\right)+\text{BesselJ}[1,\,G\,x]\right.$$
$$\left.\left.\left(\left(1-8\,G^2\,x^2\right)\,\text{BesselJ}[1,\,G\,x]+2\,G\,x\,\left(-1+2\,G^2\,x^2\right)\,\text{BesselJ}[2,\,G\,x]\right)\right)\right]$$

In[24]:=
```
G = Pi / E // N
Plot[f[x] - Imsp1b0f[x], {x, -15, 15}, PlotRange → All]
```

Out[24]=
```
1.15573
```

Out[25]=



Analytischer Test für G = 1

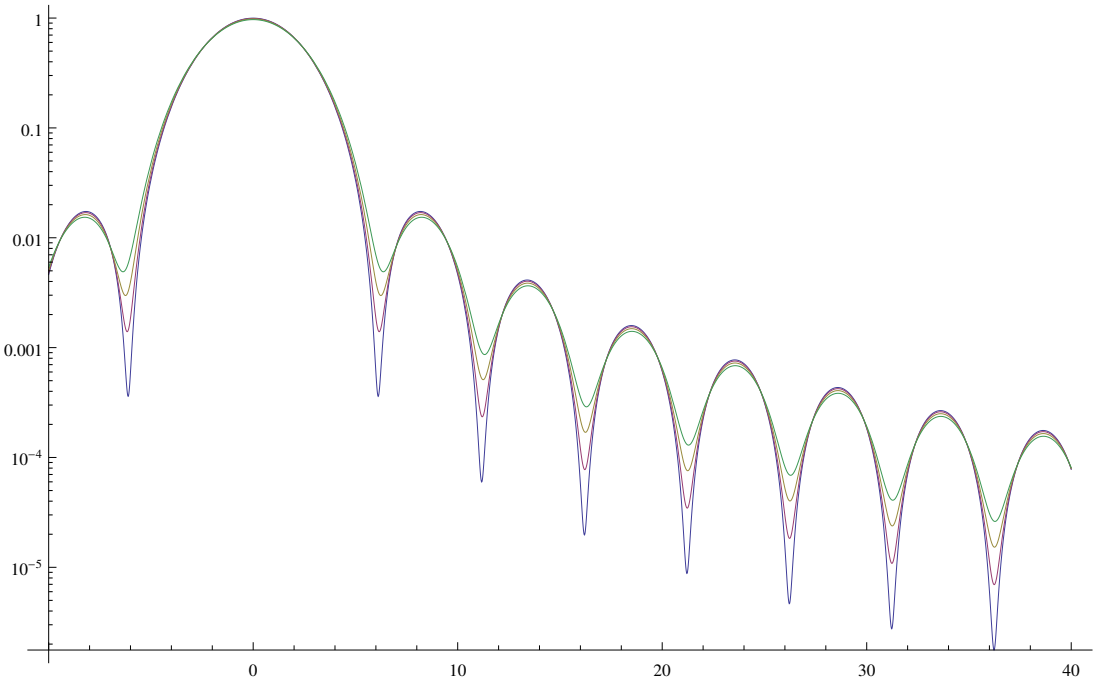
In[26]:=
```
G = 1; FullSimplify[Imsp1b0]
```

Out[26]=
$$\frac{4\,\text{BesselJ}[1,\,x]^2}{x^2}$$

Effekt von b

In[27]:=
```
G = 0.2 * Pi;
plotlist = Table[Imsp1f[x], {b, 0.25, 1.0, 0.25}];
LogPlot[plotlist, {x, -10, 40}]
```

Out[29]=



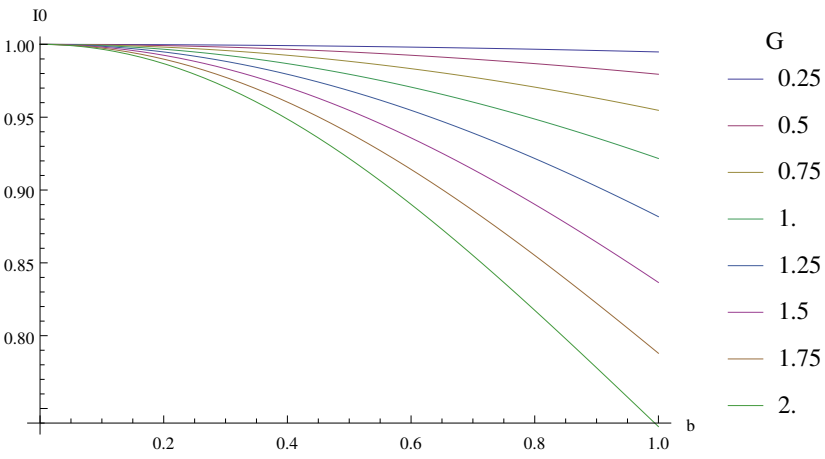
In[30]:=
```
ClearAll[G, b]
```

In[31]:=
```
IxOf = Function[{b, G}, Evaluate[IxO]]
```

Out[31]=
$$\text{Function}\left[\{b, G\}, \frac{1}{3\,b} 2 \left(4\,b\,\text{BesselJ}[0, b\,G]^2 - \frac{4\,\text{BesselJ}[0, b\,G]\,\text{BesselJ}[1, b\,G]}{G} + \frac{2\,(-1 + 2\,b^2\,G^2)\,\text{BesselJ}[1, b\,G]^2}{b\,G^2}\right)\right]$$

In[32]:=
```
plotlist = Table[IxOf[b, G], {G, 0.25, 2, 0.25}];
textlist = Table[G, {G, 0.25, 2, 0.25}];
Plot[plotlist, {b, 0, 1}, AxesLabel → {"b", "I0"},
 PlotLegends → LineLegend[textlist, LegendLabel → "G"]]
```

Out[34]=



In[35]:=
```
ClearAll[G, b]
```

In[37]:=
```
LplusImsp1 = Limit[Imsp1, x → b]
```

Out[37]=
$$\frac{1}{3\,b^2\,G^2}\left(8\,b^2\,G^2\,\text{BesselJ}[0,\,2\,b\,G]^2 - 4\,b\,G\,\text{BesselJ}[0,\,2\,b\,G]\,\text{BesselJ}[1,\,2\,b\,G] + \left(-1+8\,b^2\,G^2\right)\text{BesselJ}[1,\,2\,b\,G]^2\right)$$

In[39]:=
```
LminusImsp1 = Limit[Imsp1, x → -b]
```

Out[39]=
$$\frac{1}{3\,b^2\,G^2}\left(8\,b^2\,G^2\,\text{BesselJ}[0,\,2\,b\,G]^2 - 4\,b\,G\,\text{BesselJ}[0,\,2\,b\,G]\,\text{BesselJ}[1,\,2\,b\,G] + \left(-1+8\,b^2\,G^2\right)\text{BesselJ}[1,\,2\,b\,G]^2\right)$$

In[41]:=
```
LminusImsp1 == LplusImsp1
```

Out[41]=
```
True
```

In[43]:=
```
TeXForm[LminusImsp1]
```

Out[43]//TeXForm=
```
\frac{8 b^2 G^2 J_0(2 b G){}^2+\left(8 b^2 G^2-1\right) J_1(2 b
   G){}^2-4 b G J_1(2 b G) J_0(2 b G)}{3 b^2 G^2}
```